and to the right, as at a, must correspond to the state of an unsaturated aqueous solution of aniline, whose temperature and specific solution volume can be read, and whose osmotic pressure could be indicated by an isobaric line, had we the data for plotting it. A little thought makes it evident, too, that such isobaric lines would follow the same general course as those shown in the alcohol diagram.

Now consider what must be the effect of gradually decreasing the volume of the unsaturated vapour in the one case and the solution volume of the aniline in the unsaturated solution in the other, while temperature is kept constant. In the case of the vapour (Fig. 3), the point a will pass to the left across lines of increasing pressure, until the vapour becomes saturated at b. Then, if the diminution of volume continue, a portion of the vapour will condense to the liquid state, or be transferred to c, while the rest remains saturated vapour at b. With continued decrease of volume, the proportion condensed will constantly increase, but there can be no alteration of pressure till all is condensed; and after that nothing but a very slight diminution of volume is possible without a lowering of temperature. Well, how are we to diminish the solution volume of the aniline in the unsaturated aqueous solution? Clearly by depriving the solution of some of its water, so as to leave the same quantity of aniline distributed throughout a smaller space. And what will be the result of doing this while temperature is kept constant? Evidently, as in the other case, the point a (Fig. 2) will travel to the left, across lines of increasing osmotic pressure, until it reaches b—that is, until the solution is a saturated one; and after that, if more water be abstracted, some of the aniline will be thrown out or condensed, not as pure aniline but as a saturated solution of water in aniline, so that two layers will now co-exist—the aniline in one having the specific solution volume represented at b, and the aniline in the other having that represented at c. This tranference from b to c will continue, as water is abstracted, until the ratio of residual water to aniline is just enough to give the whole of the latter the specific solution volume shown at c. At this stage the water layer will disappear, and only a saturated solution of water in aniline will be left; and after that only a very small volume change can possibly result from further abstraction of water, as the specific solution volume is already not far from the specific volume of pure aniline itself at the same temperature.

To complete the comparison of the two curves, let me point out that, just as we can from Fig. 3 calculate the distribution of alcohol between its liquid and its vapour layers under given conditions, so can we calculate from Fig. 2 the distribution of the aniline between the aniline layer and the water layer under given conditions. In the former case, if the total volume of a tube containing n grammes of alcohol, at, say, 230°, be $n \times x$, and if x be marked off (Fig. 3) between b and c on the line of that temperature, then (x, b, and c standing for the volumes which can be read off on the horizontal base line) $n \,.\, \frac{x-c}{b-c}$ is the weight of the alcohol in the vapour layer, and $n \,.\, \frac{b-x}{b-c}$ is its weight in the liquid layer, and the volumes of the two layers in cubic centimetres are $n \,.\, b \,.\, \frac{x-c}{b-c}$ and $n \,.\, c \,.\, \frac{b-x}{b-c}$ respectively, which are together equal to $n \,.\, x$. Just so also with the aniline and water mixture (Fig. 2). If $n \times x$ be the total volume of the mixture (both layers together) containing n grammes of aniline, at, say, 140°, and if x be marked off as it was in the other case, then $n \,.\, \frac{x-c}{b-c}$ is the weight of aniline in the water layer, and $n \,.\, \frac{b-x}{b-c}$ is its weight in the aniline layer, and the total volumes of the two layers are $n \,.\, b \,.\, \frac{x-c}{b-c}$ and $n \,.\, c \,.\, \frac{b-x}{b-c}$ respectively, together equal to $n \,.\, x$.

If the actual weights of aniline and water in the mixture be given, the value of x can be calculated with a very fair approach to accuracy by the method adopted in plotting the curve; and thus all the facts with regard to the distribution at any temperature can be obtained.

Now if it be remembered that this case of aniline and water is not an isolated one, but typical of many cases experimented on by Alexéeff, and if it be remembered also that there exists no direct experimental evidence to show that the law which governs these cases is not the general law regulating all simple solutions, it must I think be granted that the facts do somewhat strongly support the hypothesis of a critical solution point which I deduced in the first instance from the general theory of solution. It may be summed up as follows:—

(1) In every system of solution which starts with a solid and its simple solvent, the solid has a solution melting-point which is lower than its true melting-point. Above this temperature the system consists of two separate liquids, each of which is a saturated solution.

(2) These two liquids become one homogeneous solution at a temperature which depends on the ratio of the original ingredients. There is one ratio which demands a higher temperature than any other. This is the critical solution temperature, above which either ingredient is infinitely soluble in the other.

THEORY OF FUNCTIONS.[1]

II.

WE now come to Dr. Schwarz's contributions to the theory of functions, which relate to the two theorems stated, and we begin with those which relate to the conform representation of various surfaces on a circle.

In the paper "Ueber einige Abbildungsaufgaben," we learn that these investigations date back to the time when the author was a student at Berlin. A fellow-student, Herr Mertens, observed to him how curious it was that Riemann had proved the existence of a function which would give a conform representation of the surface of a figure like a triangle on that of a circle, whilst the actual determination of the function in form of an expression seemed, on account of the discontinuities due to the corners, to be beyond the present powers of analysis. Dr. Schwarz thereupon tried to work out a special case, and selected the representation of a square on a circle, or, rather, first on the half of the plane which is bounded by a straight line; for the circle can be conformly represented on this half-plane by aid of reciprocal radii.

He next looks for a function which has such discontinuities as are introduced by the corners of the square. The right angle has to be represented by an angle of twice the magnitude, and therefore here the representation cannot be similar in the smallest parts. The known properties of the exponential function help at once to find it. This function is thus obtained by a happy guess, not by prescribed rules. It still contains an infinite number of constants. To determine these it is observed that if $u = f(t)$ gives a conform representation of a surface T in the plane t on a surface U in the plane u, then also will $u' = C_1 u + C_2$ gives such representation, only the corresponding figure U′ is drawn to a different scale, and placed in a different position in the u plane. On differentiating this equation with regard to

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[1] "Gesammelte Abhandlungen." Von H. A. Schwarz. Two Volumes. (Berlin: Julius Springer, 1890.) Continued from p. 323.

t, and eliminating the constants, it is found that the expression

$$\frac{d}{dt}\log\frac{du}{dt}$$

remains unaltered if we substitute u' for u, and is therefore independent of the absolute magnitude and position of the figure U in the plane. The value of this expression in the case under consideration is then shown to be a rational function of t, and on integration the value of u is got as an elliptic integral. On returning from the half-plane to the circle by the substitution $s = t - i/t + i$, we get ultimately

$$u = \int_0^s \frac{ds}{\sqrt{1-s^4}},\quad s = \sin am\, u,\quad (k = i).$$

This lemniscatic function gives the required transformation, as is ultimately easily verified.

He also gives in form of a definite integral the function for the conform representation of a triangle on a circle, and the expression found at once suggests the function for the representation of any polygon.

These results he laid, in 1864, before the Mathematical Seminary of the Berlin University. They were, however, not published till 1869, and then with considerable additions. Meanwhile the objections raised against Riemann's theory, already mentioned, had been raised. These invalidated the results about the polygon; for the formula given contains a number of constants. The complete proof for the validity of the formula requires Riemann's theorem about the existence of a solution, or else a proof that the constants can in each case be determined. As the first had become doubtful, the author now states that he had been able to give the last proof in case of any quadrilateral, and adds that he has received a general proof from Weierstrass. But this proof is not given.

A few other examples are added, viz. the representation, always on a circle, of the part of a plane outside a square, of the space bounded by a parabola and by an ellipse. This last problem is more fully considered in a separate paper.

He next considers a polygon bounded by circular arcs. Such a polygon is transformed into another also bounded by circular arcs by aid of the substitution

$$u' = (c_1 u + c_2)/(c_3 u + c_4).$$

The differential equation obtained on eliminating the four constants, is of the remarkable form $\psi(u', t) = \psi(u, t)$, where $\psi(u, t)$ is the expression which Cayley has more suitably denoted by (u, t), and called the "Schwarzian derivative," and which, through one of Cayley's formulæ, has led to Sylvester's theory of reciprocants.

Some of the results obtained are extended to representation on a sphere by aid of stereometric projection, and finally the function is given which performs the conform representation of the surface of a polyhedron on that of a sphere.

The function obtained contains, again, a number of constants. The author states that these can at once be determined in case of a regular polyhedron, but that he has not been able to prove that this can always be done.

The case of a tetrahedron is considered and fully worked out in a special paper. The result is that the surface of a tetrahedron can always be conformly represented on the surface of a sphere, so that there is a one-one correspondence between the points, and this can be done in one way only if for any three points on the one surface the corresponding ones on the other have been arbitrarily selected.

The representation of a square on a circle is illustrated by a figure. The surface of the square is divided into smaller squares by lines parallel to the sides, and the curves representing these are drawn on the circle.

As this representation is performed by elliptic functions it gives interesting illustrations of these functions. In the paper attention is called to an illustration of a theorem of Abel, and in a note, added to the present reprint, a quotation from a paper on complex prime numbers, by Jacobi, is made, of which we repeat a part here. Jacobi says:—

"Eine ebenso interessante als schwierige Aufgabe dürfte es sein, dieser Theilung des Lemniscatenbogens in $a + b\sqrt{-1}$ Theile . . . einen geometrischen Sinn abzugewinnen."

This is done by the above representation.

The chief result of the papers considered is that the possibility of a conform representation of one surface on a circle has been proved for certain cases by the actual determination of the function which gives it. Especially it has been shown that a figure bounded either by straight lines or by arcs of circles can be thus represented. These results form the starting-point for the papers in which it is attempted to give strict proofs for Riemann's general theorems stated above.

In the paper "Zur Theorie der Abbildung," it is supposed that the plane surface U to be represented on the area of a circle consists of a simply connected single sheet whose boundary is everywhere convex. The plane is covered by a net of squares, and then that area is taken which contains all those squares that lie wholly or partly within the given area. For this new area U_1, a function u_1 exists, by aid of which the area of a circle is represented on that of the new figure, for it is bounded by straight lines. On halving the sides of the squares, a new area is formed, coming nearer the given one. After m repetitions of the process, we get a function u_m, representing the area of the circle on the last-formed area U_m. If m is now increased indefinitely, then U_m will become coincident with the given area U, and it is proved that the limiting value of u_m becomes an analytical function representing the area of the circle on the given area U. This proof, however, requires that the boundary of U is everywhere convex towards the outside.

It is worthy of notice that the boundary of the surface U is replaced by a broken line whose sides always remain at right angles, and which, therefore, never changes into a curve, which has, generally speaking, at every point a definite tangent. The boundary of U_m is and remains for an increasing m a broken line. This seems closely connected with Prof. Klein's speculations about the impossibility of representing $y = f(x)$ geometrically by a curve. All that a curve can represent is what he calls a *Functionenstreifen*. If we admit this, and consider the boundary of U as given by a *Streifen* (a strip of small but not vanishing breadth), we may divide the boundary into a finite number of parts, such that the strip representing each arc contains an arc of a circle or a straight line. The figure thus obtained can be conformly represented on a circle. Thus we get the theorem that every surface bounded by a *Streifen* can be conformly represented on a circle. This is all that is wanted for physical applications. In these, therefore, there need be no hesitation of applying Riemann's theorems. In fact, physicists never have hesitated, and have used Green's and Thomson's theorems, on which Riemann's depend, as if the boundaries of the solids and surfaces considered were continuous and without thickness, though neither can be true if matter itself is discontinuous in its smallest parts, if it is made up of atoms of small, but not infinitely small, size.

Having considered the papers which deal with the conform representation of one surface on another, we come to those in which the determination of an analytical function by aid of the partial differential equation $\Delta u = 0$ is treated.

We have seen that the determination of $w = u + iv$ as an analytical function of $z = x + iy$ can be made to depend on that of its real part u as a function of x and y, which satisfies $\Delta u = 0$.

According to Riemann, such a function, u, is completely determined if it has at the boundary of a given surface any prescribed values, whilst it is finite and continuous for all points within the surface, or has given discontinuities. It is required to prove that this assertion of Riemann's is true.

In the paper "Ueber einen Grenzübergang durch alternirendes Verfahren," Schwarz gives an outline of his method, but in so short a form that it is difficult to understand the reasoning, there being constant references to theorems which can only be known to those who are well acquainted with the literature of the subject. In the next paper, however, "Ueber die Integration der partiellen Differentialgleichung $\Delta u = 0$ unter vorgeschriebenen Grenz- und Unstetigkeitsbedingungen," we have the same problem treated in an exhaustive manner. All the propositions used are first explained, and only once reference is made to a theorem which Weierstrass has given in his lectures, but not published. This paper is very clear, and contains a complete introduction to the theory of analytical function, or, as Axel Harnack says, it contains "in gedrängter Kürze eine ganze Theorie der Potential functionen" (in a plane).

It is first pointed out that our problem, of determining u subject to given boundary conditions, can always be solved for a circle, and the function which solves it is given in form of a definite integral. The proof for this statement is given in a subsequent paper, "Zur Integration der partiellen Differentialgleichung $\Delta u = 0$," which contains, also, investigations about discontinuities, &c., and which should be read together with the one now under discussion.

It is next shown that the problem for any other surface is reduced to the above if the surface can be conformly represented on the circle. This greatly enhances the importance of such representation.

Three cases of surfaces for which this representation has been obtained are enumerated, viz. a *crescent* formed by two circular arcs, including a segment of a circle; a triangle bounded by arcs of circles (or by straight lines), provided two of the angles are right angles; and, lastly, such a triangle formed by circular arcs which has the third vertex a cusp.

Next an *analytical line* is defined, viz. if $z = f(t)$ is an analytical function, then the curve which the point z describes for real values of t is said to be analytical. Such a line is therefore, by aid of the function $z = f(t)$, conformly represented on a straight line, the axis of real t.

If now the boundary of a surface consists of a finite number of pieces of analytical lines, it will be possible to get, by aid of the equation $z = f(t)$, on the plane (t) a conform representation of one of these pieces, together with a part of the surface, and this may be bounded in such a manner that its representation on the plane (t) becomes, say, a segment of a circle. For this portion of the surface, therefore, our problem (of determining u) can be solved. From the whole surface we can thus cut off pieces next to the boundary for which the problem can be solved. It will therefore also be possible to place on the surface a number of figures such that for each of them the problem can be solved, and that no part of the given surface is uncovered, whilst none projects beyond the boundary. These figures will, of course, overlap.

Suppose, now, we could prove that our problem can be solved for a surface formed by such overlapping figures, if it can be solved for each of the component figures, the part where the figures overlap counting, of course, only once; then the proof would be complete that our problem of determining u as a solution of $\Delta u = 0$, so that it has prescribed values on the boundary of a given surface, has a solution if the surface is bounded by a finite number of pieces of analytical lines.

This is what Schwarz does by the "Grenzübergang durch alternirendes Verfahren." He supposes that T_1 and T_2 denote two surfaces, such that for each of them the problem has been solved, and places these so that they overlap, and their outer boundaries form a new surface, T.

The two figures have a region T^* in common; the figure T may therefore be expressed as $T = T_1 + T_2 - T^*$. Part of the boundary of T_1 is boundary of T; the other lies within T_2 and is boundary of T^*. The same is true of T_2. There is, first, a function u_1 determined for T_1, which has, on that part of the boundary belonging to T, the prescribed value, and on the other parts, within T_2, the value zero. Then a function u_2 is determined for T_2, satisfying the analogous boundary condition, only on the part of its boundary within T_1 it receives the values which the first function u_1 has there.

Then new functions are determined alternately for T_1 and T_2, each having on that part of the boundary which belongs to T the prescribed values, and on that part which belongs to T^* the values which the last function for the other area gives it. The effect is that, ultimately, two functions, u' for T_1 and u'' for T_2, are obtained, which coincide throughout the boundary of T^*, and which, therefore, must be identical for the whole common region T^*. But this implies, again, that they are values of the same function u, which satisfies, therefore, all conditions.

Riemann's principal theorem is thus proved, not for any surface, but for a case of very great generality. For the theory of analytical functions this seems sufficient, as it is difficult to conceive the necessity of having to use surfaces bounded by quite arbitrarily given boundaries.

In the theory of potential such boundaries might occur, but here Klein's investigations already referred to are pertinent. These, in fact, seem to brush away a great number of the difficulties which have been introduced by starting from too general definitions.

The theorems used by Riemann in his theory of Abelian functions, for instance, can all be proved by the above results of Schwarz, who, indeed, shows how the discontinuities which occur here can be treated by his method. He also extends some of his results to conform representation of surfaces on a sphere, and completes the investigation in his earlier papers.

Of the remaining papers we can only give a very short account. There are first two papers on developable surfaces of the first seven orders, and more particularly on those of order five.

The papers "Bestimmung der scheinbaren Grösse eines Ellipsoids für einen beliebigen Punkt des Raumes" and "Zur conformen Abbildung der Fläche eines Rechtecks auf die Fläche einer Halbkugel" contain applications of the use of Weierstrass's elliptic functions. Those who take an interest in the latter, which are in England known chiefly through various papers by Prof. Greenhill, will be glad to learn that Herr Schwarz promises in the preface of vol. ii. a new edition of his "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen."

In the important paper "Ueber diejenigen Faelle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt," we have applications of the theories just discussed, but we must forbear from entering into a discussion of it.

The investigation rests on a discussion of the well-known differential equation of the second order, of which the series is an integral. The results obtained are included in Forsyth's "Treatise on Differential Equation," though proved in a very different manner. They have enabled Klein to determine all differential equations of the second order which have algebraical integrals.

This paper, "Ueber algebraische Isothermen," contains this theorem:—

If a complex variable $w = f(z)$ has the property that the curves for which the real part u of w has a constant value are algebraical, then w is either an algebraical function of z, or μw, where μ or μi is some real number, is the logarithm of an algebraical function, or else μw is an elliptical integral of the first kind, with real modulus whose superior limit is an algebraical function of z.

All these investigations require a deep study of the very foundations of analysis. These cannot help to reveal a number of inaccuracies and gaps in the ordinary theories as contained in text-books. Accordingly we find several papers in the collection which contain such corrections. Thus there is one paper in which a complete system of independent conditions is given which underlie the proposition that $d^2u/dxdy = d^2u/dydx$.

In another paper the definition of the area of a curved surface in the first edition of Serret's "Calcul Différentiel et Intégral" is shown to be wrong. In another long paper it is proved that of all solids of given volume the sphere has the smallest surface. All previous proofs depend on the supposition that *one solid exists* which has a minimum surface; of this the present proof is independent.

The first volume contains papers relating to surfaces of minimum area. The original problem is one of the calculus of variation, viz. a given closed curve in space being given, it is required to determine that surface bounded by it which has the least area. One of the chief properties of such surfaces is that the principal radii of curvature at each point are equal but opposite, or the mean curvature is everywhere = 0. Hence all surfaces which have this property are called "surfaces of minimum area," though it is not any longer true that surfaces of this kind have the original property for every part cut out by any curve drawn on them, just as the arc of a great circle on a sphere ceases to be the shortest line between its ends as soon as the arc becomes greater than a semicircle. The question to decide whether this is the case for a given closed curve on the surface is considered in the paper "Ueber ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung." It is, of course, a problem about the second variation.

There is an interesting connection between the surfaces of minimum area and the conform representation of one surface upon another, viz. every such representation of the whole surface of a regular polyhedron on a sphere gives rise to a surface of minimum area, which in this case contains an infinite number of straight lines.

In the first paper the surface is considered which is thus obtained from the cube. This paper, when communicated to the Berlin Academy, was illustrated by models. In the present reprint, nicely executed shaded figures of these are added. The first gives the surface corresponding to one face of the cube. It is bounded by four edges of a regular tetrahedron. In the second we have the surface corresponding to all six faces of the cube. It forms one continuous sheet. If this surface be still further continued, a surface is obtained which extends throughout the whole space. Of this the third plate gives a part. Analytically the problem depends on elliptical functions.

These investigations are continued in the second paper (and several others), which obtained a prize of the Berlin Academy. The prize problem required the complete solution, by aid of elliptical or Abelian functions, of some important problem taken from almost any part of pure or applied mathematics. Herr Schwarz treats of the surface of least area bounded by any skew quadrilateral.

In "Fortgesetzte Untersuchungen über specielle Minimalflächen," a new problem is proposed, viz. there is given a closed chain consisting of straight lines and planes, it is required to find a surface of minimum area bounded by the lines and perpendicular to the planes.

Of the other papers we mention the "Miscellen aus dem Gebiete der Minimalflächen." It contains a highly interesting review of the whole subject, including Plateau's investigations, and is full of suggestions.

The volumes, which are dedicated to Weierstrass, are well printed on octavo pages sufficiently large to give room for the formulæ required, and not so large as to be unwieldy, as is the case with a recently published "Collection." But there is one point in which the edition might have been improved, trifling as far as editing and printing are concerned, but of great benefit to the reader. It is very desirable that in all editions of collected papers the examples set by Sir William Thomson and Prof. Cayley should be followed, of placing the date of the first publication of each paper both in the table of contents and at the head of each paper.

O. HENRICI.

GEOGRAPHICAL EXPEDITIONS.

M. GROMBCHEVSKY, now at St. Petersburg, has given the Russian Geographical Society a most interesting account of his last expedition. It is known that the Expedition left Marghelan in June 1889, and that having found the Alai Mountains deeply clothed in snow, they went to Kala-i-khumb through Karategin and Vakhia. They found that the khanate of Shugnan was at war with the Afghans, and as the latter refused to let the Expedition go further, M. Grombchevsky returned to Vakhia, after having crossed the Sytarghi Pass, which has on its western slope a great glacier, six miles long. In August, after having made a long circuitous journey over the Pamir (the well-known Pamir robber, Sahir-Nazar, being the guide of the Expedition), they reached the frontier of the Pamir khanates now occupied by the Afghans, and waited there for the Ameer's permission for further advance. A refusal was received in October, when the temperature already was from 20° to 24° C. below zero, and the Expedition could find no fuel of any kind. So they crossed the Mus-tagh ridge (yaks being used for the transport of provisions), and reached the valley of the Raskem River, where they met with Mr. Younghusband. During their fifty-five days' stay on the banks of the Raskem, they explored the passes of Shimshal, Mustagh, and Balti-davan, leading to Kashmir, as well as the passes across the Raskem ridge leading to Kashgaria. In November, M. Grombchevsky was at the Kashmir fort Shahidulla-kodja; but the fort was abandoned, and, the Expedition having no provisions, they asked permission to enter Kashmir and to winter there. But Colonel Nisbet refused admission to Kashmir, so that the Expedition had nothing to do, M. Grombchevsky says, but to move eastward, across the desert plateaus of Tibet, in order to reach some inhabited spot. Moving up the Kara-kash, the Expedition ascended the Tibet plateau. The thermometer fell as low as − 33° to − 35° C., all water was frozen, and two-thirds of the horses died; so that all natural history collections were abandoned, and, notwithstanding a frightful snowstorm, the Expedition re-crossed the mountains and went to Kashgaria. The first settlements were reached in February. Next month M. Grombchevsky went to Khotan, and thence to Niya, where he met with the commander of the Tibet Expedition, M. Pyevtsoff. At the end of March, he visited the Sourgak gold-mines in the south of Niya—where he found 3000 men busy in gold-washing—and Polu, whence he again ascended the Tibet plateau, and after some explorations he returned to Kashgaria again. In the autumn he visited the middle course of the Raskem River, making acquaintance with interesting tribes of mountaineers, and thence returned to Russia. The geographical results of the expedition seem to be very important. Surveys were made over a length of 5000 miles, and latitudes and longitudes were determined at 73 different spots; heights were measured throughout the journey, and photographic